DIFFERENTIAL EQUATIONS

SELECTED FORMULAE

EXACT

Equation $M(x,y)dx + N(x,y)dy = 0$ *is exact if* $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$ *or* $M_y = N_x$ **Solution**: $F(x,y) = \int M(x,y)\partial x + \varphi(y)$

$\frac{\partial}{\partial y}F(x,y) = \frac{\partial \int M(x,y)\partial x}{\partial y} + \varphi'(y) = N(x,y)$ *solve for* $\varphi'(y)$ *and integrate* $\varphi(y) = \int \varphi'(y) + c$

■ $F(x,y) = \int M(x,y)\partial x + \varphi(y) + c$

If $M_y \neq N_x$ *an integrating factor may be* $\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$ *so* $\mu(x)M(x,y)dx + \mu(x)N(x,y)dy = 0$ *becomes exact.*

SEPARABLE

Equation $F(x)G(y)dx + f(x)g(y)dy = 0$ *is separable as* $\frac{F(x)}{f(x)}dx + \frac{G(x)}{g(x)}dy$ *or* $M(x)dx + N(x)dy = 0$

■ $F(x,y) = \int M(x)dx + \int N(y)\,dy + c$

HOMOGENEOUS

Equation $M(x,y)dx + N(x,y)dy = 0$ *is Homogeneous if* $\frac{dy}{dx} = f(x,y)$ *can be expresed as* $\frac{dy}{dx} = g(y/x)$

Solution: *set* $v = \frac{y}{x}$ $y = vx$ *and* $\frac{dy}{dx} = v + x\frac{dv}{dx}$ *so* $g(y/x) = g(v)$ *and* $v + x\frac{dv}{dx} = g(v)$ *or* $[v - g(v)]dx + xdy = 0$ *(separable)*

$\frac{dv}{v - g(v)} + \frac{dx}{x} = 0$ $\int \frac{dv}{v - g(v)} + \int \frac{dx}{x} = c$ *or* ■ $F\left(\frac{y}{x}\right) + ln|x| = c$

HOMOGENEOUS WITH CONSTANT COEFICIENTS

$a_0y''' + a_1y'' + a_2y' + a_3y = 0$ *find*: $a_0m^3 + a_1m^2 + a_2m + a_3 = 0$ *Roots*: $m = \{r_1\ r_2\ r_3\}$

■ $y = c_1e^{r_1x} + c_2e^{r_2x} + c_3e^{r_3x}$ *If* $r_2 = a + bi$ *then* $r_3 = a - bi$ *and* ■ $y = c_1e^{r_1x} + e^{ax}(c_2\sin bx + c_3\cos bx)$

UNDETERMINED COEFICIENTS (UC)

$a_0y'' + a_1y' + a_2y = F(x)$ *find the non duplicating set of UC* $\{f_1\ f_2\ f_3 \ldots\}$ $y_p = Af_1 + Bf_2 + Cf_3 \ldots$

find A B *and* C *equating* $a_0y_p'' + a_1y_p' + a_2y_p = F(x)$ *then* ■ $y = c_1e^{r_1x} + c_2e^{r_2x} + Af_1 + Bf_2 + Cf_3 \ldots$

LINEAR

An equation expressed as $\frac{dy}{dx} + P(x)y = Q(x)$ *is an ordinary linear DE.* **Solution**: ■ $e^{\int P(x)dx}y = \int e^{\int P(x)dx}Q(x)dx + c$

BERNOULLI

An equation expressed as $\frac{dy}{dx} + P(x)y = Q(x)y^n$ *is a Bernoulli DE.* **Solution**: *set* $v = y^{n-1}$ *where* $\frac{dv}{dx} = (n-1)y^{-n} \cdot \frac{dy}{dx}$

and $\frac{dv}{dx} + (n-1)P(x)v = (n-1)Q(x)$; $P_1(x) = (n-1)P(x)$; $Q_1(x) = (n-1)Q(x)$ ■ $\frac{dv}{dx} + P_1(x)v = Q_1(x)$ *(linear in v)*

ORDER REDUCTION

To reduce second order homogeneous linear DE $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$ *to first order if* $f(x)$ *is a known solution,*

set $g(x) = v(x)f(x)$ $w(x) = v'(x)$ $w(x) = \frac{e^{-\int \frac{a_1(x)}{a_0(x)}dx}}{[f(x)]^2}$ $v(x) = \int w(x)\,dx$ **General solution**: ■ $y = c_1f(x) + c_2g(x)$

Euler's formula: $e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$

CAUCHY-EULER EQUATION

Third order: $a_0x^3y''' + a_1x^2y'' + a_2xy' + a_3y = F(x)$ *substitute:* $x = e^t$ *then* $t = \ln x$ $x^3y''' = y_t''' - 3y_t'' + 2y_t'$ $x^2y'' = y_t'' - y_t'$

$xy' = y_t'$ **Charac. eq:** $a_0m^3 + a_1m^2 + a_2m + a_3 = 0$ **Roots:** $m = \{r_1,\ r_2,\ r_3\}$ $y_{tc} = c_1e^{r_1t} + c_2e^{r_2t} + c_3e^{r_3t}$

determine $F(t)$ *particular integral coef. and convert* $y(t)$ *to* ■ $y(x) = c_1y_1(x) + c_2y_2(x) + c_3y_3(x) + Af(x)_1 + Bf(x)_2 + \cdots$

VARIATION OF PARAMETERS

Second order: $a_0(x)y'' + a_1(x)y' + a_2(x)y = F(x)$ $y_c = c_1y_1(x) + c_2y_2(x)$ $y_p = v_1(x)y_1(x) + v_2(x)y_2(x)$

$\begin{vmatrix} v_1' \\ v_2' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}^{-1} \times \begin{vmatrix} 0 \\ F \end{vmatrix}$ $v_1 = \int v_1'\,dx$ $v_2 = \int v_2'\,dx$ ■ $y = c_1y_1(x) + c_2y_2(x) + v_1(x)y_1(x) + v_2(x)y_2(x)$

LAPLACE TRANSFORM

$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}f(t)dt$ $f(t) = \mathcal{L}^{-1}\{F(s)\}$ $\mathcal{L}\{y(t)\} = Y(s)$ $\mathcal{L}\{y'\} = sY - y(0)$ $\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)$

$\mathcal{L}\{y'''\} = s^3Y - s^2y(0) - sy'(0) - y''(0)$ $a_0y'' + a_1y' + a_2y = F(x)$ $a_0\mathcal{L}\{y''\} + a_1\mathcal{L}\{y'\} + a_2\mathcal{L}\{y\} = \mathcal{L}\{F\}$

$a_0\left(s^2Y - sy(0) - y'(0)\right) + a_1\left(sY - y(0)\right) + a_2Y = \mathcal{L}\{F\}$ *solve for* $Y(s)$ *and find* ■ $y = \mathcal{L}^{-1}\{Y(s)\}$

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	1	$\frac{1}{s}$
2	e^{at}	$\frac{1}{s-a}$
3	$\sin bt$	$\frac{b}{s^2+b^2}$
4	$\cos bt$	$\frac{s}{s^2+b^2}$
5	$\sinh bt$	$\frac{b}{s^2-b^2}$
6	$\cosh bt$	$\frac{s}{s^2+b^2}$
7	t^n	$\frac{n!}{s^{n+1}}$
8	t^ne^{at}	$\frac{n!}{(s-a)^{n+1}}$

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
9	$t\sin bt$	$\frac{2bs}{(s^2+b^2)^2}$
10	$t\cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
11	$e^{-at}\sin bt$	$\frac{b}{(s+a)^2+b^2}$
12	$e^{-at}\cos bt$	$\frac{s+a}{(s+a)^2-b^2}$
13	$\frac{\sin bt - bt\cos bt}{2b^3}$	$\frac{1}{(s^2+b^2)^2}$
14	$\frac{t\sin bt}{2b}$	$\frac{s}{(s^2+b^2)^2}$
15	$u_a(t)$	$\frac{e^{-as}}{s}$
16	$u_a(t)f(t-a)$	$e^{-as}F(s)$